29TH JUNIOR BALKAN MATHEMATICAL OLYMPIAD

26th June, 2025 *Language: English*

Problem 1. For all positive real numbers a, b, c, prove that

$$\frac{(a^2+bc)^2}{b+c}+\frac{(b^2+ca)^2}{c+a}+\frac{(c^2+ab)^2}{a+b}\geq\frac{2abc(a+b+c)^2}{ab+bc+ca}\,.$$

Problem 2. Determine all numbers of the form

$$20252025\ldots2025$$

(consisting of one or more consecutive blocks of 2025) that are perfect squares of positive integers.

Problem 3. Let ABC be a right-angled triangle with $\angle A=90^\circ$, let D be the foot of the altitude from A to BC, and let E be the midpoint of DC. The circumcircle of $\triangle ABD$ intersects AE again at point F. Let X be the intersection of the lines AB and DF. Prove that $XD=XC$.

Problem 4. Let n be a positive integer. The integers from 1 to n are written in the cells of an $n\times n$ table (one integer per cell) so that each of them appears exactly once in each row and exactly once in each column. Denote by r_i the number of pairs (a,b) of numbers in the i^{th} row $(1\leq i\leq n)$, such that $a>b$, but a is written to the left of b (not necessarily next to it). Denote by c_j the number of pairs (a,b) of numbers in the j^{th} column $(1\leq j\leq n)$, such that $a>b$, but a is written above b (not necessarily next to it). Determine the largest possible value of the sum

$$r_1+r_2+\cdots+r_n+c_1+c_2+\cdots+c_n\,.$$

Note: In the $n\times n$ table we label the rows 1 to n from top to bottom, and we label the columns 1 to n from left to right.

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Time: 4 hours and 30 minutes.
Every problem is worth 10 points.